

# Characterization of atomic decompositions, Banach frames, $X_d$ -frames, duals and synthesis-pseudo-duals, with application to Hilbert frame theory

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## Abstract

In this paper we give a characterization of atomic decompositions, Banach frames,  $X_d$ -Riesz bases,  $X_d$ -frames,  $X_d$ -Bessel sequences, sequences satisfying the lower  $X_d$ -frame condition, duals of  $X_d$ -frames and synthesis pseudo-duals, based on an operator acting on the canonical basis of a sequence space. We also consider necessary and sufficient conditions for operators to preserve the sequence type of the listed concepts. Further, we discuss relationships between expansions in  $X$  and  $X^*$ . Finally, we apply some of the results to problems in Hilbert frame theory.

Keywords: frame, atomic decomposition, Banach frame,  $X_d$ -frame,  $X_d$ -Bessel sequence, lower  $X_d$ -frame condition,  $X_d$ -Riesz basis, dual, synthesis-pseudo-dual

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# 1 Introduction and basic definitions

The frame-concept was introduced by Duffin and Schaeffer [11] in 1952. The sequence  $(g_i)_{i=1}^\infty$  is called a (Hilbert) frame for the Hilbert space  $\mathcal{H}$  with bounds  $A, B$  if  $A$  and  $B$  are positive constants and  $A\|h\|^2 \leq \sum_{i=1}^\infty |\langle h, g_i \rangle|^2 \leq B\|h\|^2$  for every  $h \in \mathcal{H}$ . It took many years till the importance of frames was realized. Around 1990, the frame-theory began to develop in connection with Gabor analysis and wavelets [14, 12, 13]. Nowadays, frames are very important both for theory and life-applications. For more on frame-theory see [4, 8, 21, 22]. What makes frames very useful is that they require less restrictive conditions on the sequence elements compare to orthonormal bases and still they allow reconstructions of the space elements. If  $(g_i)_{i=1}^\infty$  is a frame for  $\mathcal{H}$ , then there exists a frame  $(f_i)_{i=1}^\infty$  for  $\mathcal{H}$  so that

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i, \quad \forall f \in \mathcal{H}, \quad (1)$$

and

$$g = \sum_{i=1}^{\infty} \langle g, f_i \rangle g_i, \quad \forall g \in \mathcal{H}; \quad (2)$$

such a frame  $(f_i)_{i=1}^\infty$  is called a *dual frame* of  $(g_i)_{i=1}^\infty$ . It is of importance that *overcomplete frames* (i.e. frames which are not Schauder bases) have infinitely many dual frames. For example, while the canonical dual of a Gabor frame is also a Gabor frame, in [15] one can see an example of a wavelet system for which the canonical dual does not have the wavelet structure, but there exist other dual frames with the wavelet structure. Characterizations of the dual frames of a given frame can be found in [6, 8, 24, 25].

There exist overcomplete frames  $(g_i)_{i=1}^\infty$  such that both expansions (1) and (2) hold via a sequence  $(f_i)_{i=1}^\infty$  which is not a frame (consider, for example, the frame  $(g_i)_{i=1}^\infty = (\frac{1}{2}e_1, e_2, \frac{1}{2^2}e_1, e_3, \frac{1}{2^3}e_1, e_4, \dots)$  and the sequence  $(f_i)_{i=1}^\infty = (e_1, e_2, e_1, e_3, e_1, e_4, \dots)$ , where  $(e_i)_{i=1}^\infty$  denotes an orthonormal basis for  $\mathcal{H}$ ). Furthermore, there exist overcomplete frames  $(g_i)_{i=1}^\infty$  and non-Bessel sequences  $(f_i)_{i=1}^\infty$  so that (1) holds and (2) does not hold, see Example 3.5. In the present paper we characterize all the sequences  $(f_i)_{i=1}^\infty$  satisfying (1) for a given frame  $(g_i)_{i=1}^\infty$  and call them *synthesis-pseudo duals* (in short, *s-pseudo-duals*) of  $(g_i)_{i=1}^\infty$ . For investigation of series expansions in a subspace  $\mathcal{H}_0$  of  $\mathcal{H}$  via a frame for  $\mathcal{H}_0$  and a sequence which does not necessarily belong to  $\mathcal{H}_0$ , we refer to [18].

A natural extension of the frame inequalities to Banach spaces leads to the concepts  $p$ -frame [2] and  $X_d$ -frame [9] (see Definition 1.1). In contrast to the frame-case, the  $X_d$ -frame inequalities do not necessarily lead to reconstruction via series expansions. With aim to have reconstructions, *atomic decompositions* (giving reconstruction via series expansions) were considered by Feichtinger and Gröchenig [16, 17, 20] and the concept of *Banach frame* (giving reconstruction via an operator) was introduced in [20]. Historically, first atomic decompositions and Banach frames were introduced, and after that the concepts  $p$ -frame and  $X_d$ -frame appeared.

**Definition 1.1** *Let  $X$  be a Banach space,  $X_d$  be a BK-space (i.e., a Banach sequence space for which the coordinate functionals are continuous) and  $(g_i)_{i=1}^\infty \in (X^*)^\mathbb{N}$ . The sequence  $(g_i)_{i=1}^\infty$  is called a Banach frame for  $X$  with respect to  $X_d$  if*

- (i)  $(g_i(f))_{i=1}^\infty \in X_d, \forall f \in X$ ;
- (ii)  $\exists$  positive constants  $A$  and  $B$  so that  $A\|f\|_X \leq \|(g_i(f))_{i=1}^\infty\|_{X_d} \leq B\|f\|_X, \forall f \in X$ ;
- (iii)  $\exists$  bounded operator  $Q : X_d \rightarrow X$  so that  $Q(g_i(f))_{i=1}^\infty = f, \forall f \in X$ .

*Let  $(f_i)_{i=1}^\infty \in X^\mathbb{N}$ . The pair  $((g_i)_{i=1}^\infty, (f_i)_{i=1}^\infty)$  is called an atomic decomposition of  $X$  with respect to  $X_d$  if (i) and (ii) hold and*

$$(iii') \quad f = \sum_{i=1}^\infty g_i(f) f_i, \quad \forall f \in X.$$

*The sequence  $(g_i)_{i=1}^\infty$  is called an  $X_d$ -frame (resp.  $X_d$ -Bessel sequence) for  $X$  if (i) and (ii) (resp (i) and the upper inequality in (ii)) hold.*

*It is said that  $(g_i)_{i=1}^\infty$  satisfies the lower  $X_d$ -frame condition if the lower inequality in (ii) holds for all those  $f$  for which  $(g_i(f))_{i=1}^\infty \in X_d$ .*

*When  $X_d = \ell^p$ , an  $X_d$ -frame is called a  $p$ -frame.*

In the present paper we characterize all the concepts from the above definition. When  $X_d = \ell^2$  and  $X$  is a Hilbert space, characterizations of the corresponding concepts can be found in [8, 3] and references therein. Note that if  $X_d = \ell^2$  and  $X$  is a Hilbert space, then  $X_d$ -frame, Banach frame and the first sequence in an atomic decomposition mean the same (namely, a Hilbert frame). For general Banach spaces, these three types of sequences are not the same, for a detail discussion about their relationship and differences see [31]. Recall that Hilbert frames which are at the same time Schauder bases

are called Riesz bases. Riesz bases were generalized to Banach spaces under the name of  $X_d$ -Riesz bases. The concept of  $X_d$ -Riesz basis was established by Feichtinger and Zimmermann [19]. Another definition for an  $X_d$ -Riesz basis is considered in [28], motivated by the definitions of a  $p$ -Riesz basis in [2, 10]. When  $X_d$  has the canonical vectors as a Schauder basis, the definitions in [19] and [28] are equivalent.

**Definition 1.2** *Let  $Y$  be a Banach space and  $X_d$  be a Banach sequence space. A sequence  $(g_i)_{i=1}^\infty \in Y^\mathbb{N}$  is called an  $X_d$ -Riesz basis for  $Y$  with bounds  $A, B$ , if it is complete in  $Y$ , the constants  $A$  and  $B$  are positive, and*

$$A \|(c_i)_{i=1}^\infty\|_{X_d} \leq \left\| \sum_{i=1}^\infty c_i g_i \right\|_Y \leq B \|(c_i)_{i=1}^\infty\|_{X_d}, \quad \forall (c_i)_{i=1}^\infty \in X_d. \quad (3)$$

The paper is organized as follows. Section 2 concerns the notation used in the paper. In Section 3 we give a characterization of the sequences defined in 1.1 and 1.2 based on an operator acting on the canonical basis of the corresponding sequence space. Given an  $X_d$ -frame  $\mathbb{G}$  for a Banach space  $X$ , we then characterize all the sequences  $\mathbb{F}$  which give atomic decompositions  $(\mathbb{G}, \mathbb{F})$  of  $X$  with respect to  $X_d$ ; among those  $\mathbb{F}$ , we characterize the ones which have “dual” properties, namely, which are  $X_d^*$ -frames for  $X^*$ . Further, we discuss connections between expansions in  $X$  and  $X^*$ . When  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ , validity of the representations  $f = \sum_{i=1}^\infty g_i(f) f_i$ ,  $\forall f \in X$ , is not equivalent to validity of the representations  $g = \sum_{i=1}^\infty g(f_i) g_i$ ,  $\forall g \in X^*$ , even in the Hilbert-frame setting (see Example 3.5). If  $f = \sum_{i=1}^\infty g_i(f) f_i$ ,  $\forall f \in X$ , we determine a subset of  $X^*$  where the representation  $g = \sum_{i=1}^\infty g(f_i) g_i$  holds. Section 4 concerns operators which preserve the sequence type. We determine necessary and sufficient conditions, discussing the case of bounded operators and the case of not necessarily bounded ones. Applying results from Section 3, in Section 5 we solve some problems in Hilbert frame theory. In particular, we characterize all the synthesis-pseudo-duals of a Hilbert frame.

## 2 Notation and Preliminaries

Throughout the paper,  $X$  denotes a separable Banach space and  $X^*$  denotes its dual;  $X_d$  denotes a Banach sequence space and  $X_d^*$  denotes its dual;  $\mathcal{H}$  denotes an infinite-dimensional Hilbert space and  $(e_i)_{i=1}^\infty$  denotes an orthonormal basis of  $\mathcal{H}$ . The letter  $\mathbb{G}$  (resp.  $\mathbb{F}$ ) means a sequence  $(g_i)_{i=1}^\infty$  (resp.  $(f_i)_{i=1}^\infty$ ) with elements from  $X^*$  (resp.  $X$ ). A linear mapping is called an *operator*.

The domain (resp., the range) of an operator  $V$  is denoted by  $\mathcal{D}(V)$  (resp.,  $\mathcal{R}(V)$ ). The space  $X_d$  is called a *BK-space* if the coordinate functionals are continuous. If the canonical vectors form a Schauder basis for  $X_d$ , then  $X_d$  is called a *CB-space* and the canonical basis is denoted by  $(\delta_i)_{i=1}^\infty$ . In particular, the canonical basis of  $\ell^2$  is denoted by  $(\delta_i)_{i=1}^\infty$ . When  $X_d$  is a *CB-space*, then  $X_d^\circledast := \{(V\delta_i)_{i=1}^\infty : V \in X_d^*\}$  with the norm  $\|(V\delta_i)_{i=1}^\infty\|_{X_d^\circledast} := \|V\|_{X_d^*}$  is a *BK-space* isometrically isomorphic to  $X_d^*$  [23] and for the rest of the paper  $X_d^*$  is identified with  $X_d^\circledast$ . If  $X_d$  is reflexive and a *CB-space*, then  $X_d$  is called an *RCB-space*. When  $X_d$  is an *RCB-space*, then  $X_d^\circledast$  is a *CB-space* and its canonical basis is denoted by  $(\delta_i^*)_{i=1}^\infty$ . The linear span of some elements  $x_i$  from  $X$ ,  $i \in \mathbb{N}$ , is denoted by  $\text{lin}\{x_i\}_{i=1}^\infty$ . The abbreviation ‘wrt’ stands for ‘with respect to’.

For given *CB-space*  $X_d$  and sequence  $\mathbb{G} \in (X^*)^\mathbb{N}$ , being an  $X_d$ -Bessel sequence for  $X$  or satisfying the lower  $X_d$ -frame condition, the *analysis operator*  $U_\mathbb{G}$  and the *synthesis operator*  $T_\mathbb{G}$  are given by

$$U_\mathbb{G} : \mathcal{D}(U_\mathbb{G}) \rightarrow X_d, \quad U_\mathbb{G}f = (g_i(f))_{i=1}^\infty, \quad (4)$$

$$T_\mathbb{G} : \mathcal{D}(T_\mathbb{G}) \rightarrow X^*, \quad T_\mathbb{G}(c_i)_{i=1}^\infty = \sum_{i=1}^\infty c_i g_i, \quad (5)$$

where  $\mathcal{D}(U_\mathbb{G}) = \{f \in X : (g_i(f))_{i=1}^\infty \in X_d\}$  and  $\mathcal{D}(T_\mathbb{G}) = \{(c_i)_{i=1}^\infty \in X_d^* : \sum_{i=1}^\infty c_i g_i \text{ converges in } X^*\}$ .

Note that the definition of an  $X_d$ -Bessel sequence  $\mathbb{G}$  for  $X$  requires  $\mathcal{D}(U_\mathbb{G}) = X$ , while the definition of a sequence satisfying the lower  $X_d$ -frame condition allows the domain of its analysis operator to be a subset of  $X$ .

If  $X_d$  is a *BK-space* and  $\mathbb{G}$  satisfies the lower  $X_d$ -frame condition, then  $\mathcal{R}(U_\mathbb{G})$  is closed in  $X_d$  and  $U_\mathbb{G}$  has a bounded inverse  $U_\mathbb{G}^{-1} : \mathcal{R}(U_\mathbb{G}) \rightarrow \mathcal{D}(U_\mathbb{G})$  [30].

If  $(g_i)_{i=1}^\infty$  is a (Hilbert) frame for  $\mathcal{H}$  and  $(f_i)_{i=1}^\infty$  satisfies (1) or (2), then  $(f_i)_{i=1}^\infty$  satisfies the lower frame condition [5, 25]. In Banach spaces the corresponding result is as follows:

**Lemma 2.1** *Assume that  $X_d$  is a *CB-space*. Let  $\mathbb{G}$  be an  $X_d$ -Bessel sequence for  $X$  and let there exist  $\mathbb{F}$  so that  $f = \sum_{i=1}^\infty g_i(f)f_i$  for every  $f \in X$  or  $g = \sum_{i=1}^\infty g(f_i)g_i$  for every  $g \in X^*$ . Then  $\mathbb{F}$  satisfies the lower  $X_d^*$ -frame condition, i.e., there exists a positive constant  $A$  so that  $A\|g\|_{X^*} \leq \|(g(f_i))_{i=1}^\infty\|_{X_d^*}$  for those  $g$  for which  $(g(f_i))_{i=1}^\infty \in X_d^*$ .*

### 3 Characterization of the sequences based on an operator acting on the canonical basis

As it was already mentioned in the Introduction, the concept of (Hilbert) frame is extended to three different concepts in Banach spaces, namely, atomic decompositions, Banach frames, and  $X_d$ -frames. Here we characterize each one of them, as well as  $X_d$ -Riesz bases and sequences which may fail the upper or the lower  $X_d$ -frame condition. Note that the roles of  $\mathbb{G}$  and  $\mathbb{F}$  in an atomic decomposition  $(\mathbb{G}, \mathbb{F})$  are not symmetric, so we characterize each one of them.

**Theorem 3.1** *Let  $X_d$  be an RCB-space and let  $X$  be reflexive.*

- (i) *The  $X_d$ -Bessel sequences for  $X$  are precisely the sequences  $(T\delta_i^*)_{i=1}^\infty$ , where  $T : X_d^* \rightarrow X^*$  is a bounded operator.*
- (ii) *The sequences in  $(X^*)^\mathbb{N}$  satisfying the lower  $X_d$ -frame condition are precisely the sequences  $(T\delta_i^*)_{i=1}^\infty$ , where the operator  $T : \mathcal{D}(T)(\subseteq X_d^*) \rightarrow X^*$  satisfies the properties:  $\mathcal{D}(T) \supseteq \text{lin}\{\delta_i^*\}_{i=1}^\infty$ ,  $T(\sum_{i=1}^n c_i \delta_i^*) \rightarrow T(\sum_{i=1}^\infty c_i \delta_i^*)$  as  $n \rightarrow \infty$  for every  $\sum_{i=1}^\infty c_i \delta_i^* \in \mathcal{D}(T)$ , and there exists  $\lambda \in (0, \infty)$  so that  $\|T^*(F)\| \geq \lambda\|F\|$  for every  $F \in \mathcal{D}(T^*)$ .*
- (iii) *The  $X_d$ -frames for  $X$  are precisely the sequences  $(T\delta_i^*)_{i=1}^\infty$ , where  $T : X_d^* \rightarrow X^*$  is a bounded surjective operator.*
- (iv) *The Banach frames for  $X$  wrt  $X_d$  are precisely the sequences  $(T\delta_i^*)_{i=1}^\infty$ , where  $T : X_d^* \rightarrow X^*$  is a bounded surjective operator which has a bounded right inverse defined from  $X^*$  into  $X_d^*$ .*
- (v) *The sequences  $\mathbb{G}$  for which there exists an atomic decomposition  $(\mathbb{G}, \mathbb{F})$  for  $X$  wrt  $X_d$  are precisely the sequences  $(T\delta_i^*)_{i=1}^\infty$ , where  $T : X_d^* \rightarrow X^*$  is a bounded surjective operator such that  $T^*$  has a left inverse  $L : \mathcal{D}(L)(\subseteq X_d) \rightarrow X$  satisfying  $\mathcal{D}(L) \supseteq \text{lin}\{\delta_i\} \cup \mathcal{R}(T^*)$  and  $L(\sum_{i=1}^n T\delta_i^*(f)\delta_i) \rightarrow L(\sum_{i=1}^\infty T\delta_i^*(f)\delta_i)$  as  $n \rightarrow \infty$  for every  $f \in X$ .*
- (vi) *The sequences  $\mathbb{F}$  for which there exists an atomic decomposition  $(\mathbb{G}, \mathbb{F})$  for  $X$  wrt  $X_d$  are precisely the sequences  $(T\delta_i)_{i=1}^\infty$ , where the operator  $T : \mathcal{D}(T)(\subseteq X_d) \rightarrow X$  satisfies the properties:  $\mathcal{D}(T) \supseteq \text{lin}\{\delta_i\}$ ,  $T(\sum_{i=1}^n c_i \delta_i) \rightarrow T(\sum_{i=1}^\infty c_i \delta_i)$  as  $n \rightarrow \infty$  for every  $\sum_{i=1}^\infty c_i \delta_i \in \mathcal{D}(T)$  and  $T$  has a bounded right inverse  $U : X \rightarrow X_d$  with  $\mathcal{R}(U) \subseteq \mathcal{D}(T)$  and with surjective adjoint  $U^*$ .*

(vii) The  $X_d$ -Riesz bases for  $X$  are precisely the sequences  $(T\delta_i)_{i=1}^\infty$ , where  $T : X_d \rightarrow X$  is a bounded bijective operator.

**Proof.** (i) follows from [9, Cor. 3.3].

(ii) First assume that the operator  $T : \mathcal{D}(T) (\subseteq X_d^*) \rightarrow X^*$  satisfies the conditions listed in (ii) and consider the sequence  $\mathbb{G}$  given by  $g_i = T\delta_i^*, i \in \mathbb{N}$ . Let  $f \in X$  be such that  $(g_i(f)) \in X_d$  and let  $F$  denote its correspondent element in  $X^{**}$ . Consider an arbitrary element  $\sum_{i=1}^\infty c_i \delta_i^* \in \mathcal{D}(T)$ . Let  $C$  denote the basis constant of the canonical basis of  $X_d$ . For every  $n \in \mathbb{N}$ ,

$$\left| FT \left( \sum_{i=1}^n c_i \delta_i^* \right) \right| = \left| \sum_{i=1}^n c_i g_i(f) \right| \leq C \left\| \sum_{i=1}^n c_i \delta_i^* \right\| \left\| \sum_{i=1}^\infty g_i(f) \delta_i \right\|.$$

Taking limit when  $n \rightarrow \infty$ , it follows that  $FT$  is bounded on  $\mathcal{D}(T)$ . Therefore,  $F$  belongs to  $\mathcal{D}(T^*)$ . Furthermore,

$$\|(g_i(f))_{i=1}^\infty\| = \|T^*(F)\| \geq \lambda \|f\|,$$

which completes the proof.

Conversely, assume that  $\mathbb{G}$  satisfies the lower  $X_d$ -frame condition. Then the operator  $T = T_{\mathbb{G}}$  has the required properties.

(iii) follows from [30, Th. 3.9].

(iv) Let  $\mathbb{G}$  be a Banach frame for  $X$  w.r.t.  $X_d$  and let  $Q$  denote a Banach frame operator for  $\mathbb{G}$ . By (iii), the operator  $T_{\mathbb{G}} : X_d^* \rightarrow X^*$  is bounded and surjective. Furthermore,  $Q^*$  is a bounded right inverse of  $T_{\mathbb{G}}$ . Take  $T = T_{\mathbb{G}}$ .

Conversely, assume that  $T : X_d^* \rightarrow X^*$  is a bounded surjective operator which has a bounded right inverse  $W : X^* \rightarrow X_d^*$ . Consider the sequence  $(g_i)_{i=1}^\infty = (T\delta_i^*)_{i=1}^\infty$ . By (iii),  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ . Furthermore,  $W^*$  is a Banach frame operator for  $\mathbb{G}$ .

(v) Let  $\mathbb{G}$  be such that there exists an atomic decomposition  $(\mathbb{G}, \mathbb{F})$  for  $X$  wrt  $X_d$ . By (iii),  $T_{\mathbb{G}}$  is bounded and surjective. Consider the operator

$$T_{\mathbb{F}} : \mathcal{D}(T_{\mathbb{F}}) \rightarrow X, \quad T_{\mathbb{F}}((c_i)_{i=1}^\infty) = \sum_{i=1}^\infty c_i f_i, \quad (6)$$

$$\text{where } \mathcal{D}(T_{\mathbb{F}}) = \left\{ (c_i)_{i=1}^\infty \in X_d : \sum_{i=1}^\infty c_i f_i \text{ converges in } X \right\}. \quad (7)$$

Then we have  $R(T_{\mathbb{G}}^*) = R(U_{\mathbb{G}}) \subseteq \mathcal{D}(T_{\mathbb{F}})$ . Clearly,  $\text{lin}\{\delta_i\} \subseteq \mathcal{D}(T_{\mathbb{F}})$ . Furthermore, for every  $f \in X$ ,  $T_{\mathbb{F}}(\sum_{i=1}^n g_i(f) \delta_i) \xrightarrow{n \rightarrow \infty} T_{\mathbb{F}}(\sum_{i=1}^\infty g_i(f) \delta_i)$  and  $T_{\mathbb{F}} T_{\mathbb{G}}^*(f) = \sum_{i=1}^\infty g_i(f) f_i = f$ . Take  $T = T_{\mathbb{G}}$  and  $L = T_{\mathbb{F}}$ .

Conversely, assume that the operator  $T$  satisfies the conditions listed in (v) and consider the sequence  $\mathbb{G} = (T\delta_i^*)_{i=1}^\infty$ . By (iii),  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ . Define  $f_i := L\delta_i$ ,  $i \in \mathbb{N}$ . For every  $f \in X$ ,

$$\sum_{i=1}^n g_i(f)f_i = L \left( \sum_{i=1}^n g_i(f)\delta_i \right) \xrightarrow{n \rightarrow \infty} L \left( \sum_{i=1}^\infty g_i(f)\delta_i \right) = LT^*(f) = f,$$

which completes the proof.

(vi) Let  $\mathbb{F}$  be such that an atomic decomposition  $(\mathbb{G}, \mathbb{F})$  for  $X$  wrt  $X_d$  exists. By Lemma 2.1,  $\mathbb{F}$  satisfies the lower  $X_d^*$ -frame condition. Consider the operator  $T_{\mathbb{F}}$  given by (6) and (7). Clearly,  $\mathcal{D}(T_{\mathbb{F}}) \supseteq \text{lin}\{\delta_i\}_{i=1}^\infty$ . Using (ii), it follows that  $T_{\mathbb{F}}(\sum_{i=1}^n c_i \delta_i) \rightarrow T_{\mathbb{F}}(\sum_{i=1}^\infty c_i \delta_i)$  as  $n \rightarrow \infty$  for every  $\sum_{i=1}^\infty c_i \delta_i \in \mathcal{D}(T_{\mathbb{F}})$ . Since  $(g_i)_{i=1}^\infty$  is an  $X_d$ -frame for  $X$ , it follows that  $U_{\mathbb{G}}$  is bounded with bounded inverse on  $\mathcal{R}(U_{\mathbb{G}})$  which implies (see, e.g., [26, Th. 4.15]) that  $U_{\mathbb{G}}^*$  is surjective. Furthermore, we have  $\mathcal{R}(U_{\mathbb{G}}) \subseteq \mathcal{D}(T_{\mathbb{F}})$  and  $U_{\mathbb{G}}$  is a right inverse of  $T_{\mathbb{F}}$ . Take  $T = T_{\mathbb{F}}$  and  $U = U_{\mathbb{G}}$ .

Conversely, assume that  $T$  satisfies the conditions listed in (vi) and consider the sequence  $f_i = T\delta_i$ ,  $i \in \mathbb{N}$ . Define  $g_i := U^*\delta_i^*$ ,  $i \in \mathbb{N}$ . By (iii),  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ . For every  $f \in X$ , we have that  $(g_i(f))_{i=1}^\infty = Uf \in \mathcal{R}(U) \subseteq \mathcal{D}(T)$  and

$$\sum_{i=1}^n g_i(f)f_i = T \left( \sum_{i=1}^n g_i(f)\delta_i \right) \rightarrow T U f = f,$$

which completes the proof.

(vii) follows from [28, Prop. 3.4].  $\square$

## Characterization of duals of $X_d$ -frames

While a Hilbert frame  $\mathbb{G}$  always has a sequence  $\mathbb{F}$  satisfying (1), this is not so in the general case of an  $X_d$ -frame. Casazza has proved that there exist Banach spaces  $X$  and  $p$ -frames  $\mathbb{G}$  for  $X$  such that there is no sequence  $\mathbb{F}$  satisfying

$$f = \sum_{i=1}^\infty g_i(f)f_i, \forall f \in X. \quad (8)$$

The sequence  $(g_i)_{i=1}^\infty = (e_i + e_{i+1})_{i=1}^\infty$  is an  $X_d$ -frame (even a Banach frame) for  $\mathcal{H}$  with respect to appropriate sequence space  $X_d$  and there is no sequence  $(f_i)_{i=1}^\infty \in \mathcal{H}^\mathbb{N}$  such that  $f = \sum_{i=1}^\infty \langle f, g_i \rangle f_i$  holds for all  $f \in \mathcal{H}$  [9]. Theorem 3.1(v) gives a characterization of the  $X_d$ -frames  $\mathbb{G}$  for which there exists  $\mathbb{F}$  satisfying (8).



By [9], when  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ ,  $X_d$  is an  $RCB$ -space and  $\mathcal{R}(U_{\mathbb{G}})$  is complemented in  $X_d$ , then there exists  $\mathbb{F}$  which is an  $X_d^*$ -frame for  $X^*$  and satisfies (8); such a sequence  $\mathbb{F}$  is called a *dual  $X_d^*$ -frame of  $\mathbb{G}$* . There exist  $X_d$ -frames  $\mathbb{G}$  for  $X$  and sequences  $\mathbb{F}$  satisfying (8) but not necessarily being  $X_d^*$ -frames for  $X^*$  (see Example 3.2). Given an  $X_d$ -frame  $\mathbb{G}$  for  $X$ , Theorem 3.3 gives a characterization of all the sequences  $\mathbb{F}$  satisfying (8) (such  $\mathbb{F}$  will be called *synthesis-pseudo-duals* of  $\mathbb{G}$ , in short, *s-pseudo-duals* of  $\mathbb{G}$ ) if such ones exist, and a characterization of those ones among them, which are  $X_d^*$ -frames for  $X^*$ .

**Example 3.2** [30, Ex. 5.1] *Let  $X = \ell^p$  ( $1 < p < \infty$ ),  $(\xi_i)_{i=1}^\infty$  be the canonical basis of  $X$  and  $(E_i)_{i=1}^\infty$  be the coefficient functionals associated to  $(\xi_i)_{i=1}^\infty$ . Then the sequence  $\mathbb{G} = (\frac{1}{2}E_1, E_2, \frac{1}{2^2}E_1, E_3, \frac{1}{2^3}E_1, E_4, \dots)$  is a  $p$ -frame for  $X$ , the sequence  $\mathbb{F} = (\xi_1, \xi_2, \xi_1, \xi_3, \xi_1, \xi_4, \dots)$  satisfies the lower  $q$ -frame condition, but does not satisfy the upper one, and*

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \quad \forall f \in X, \quad \text{and} \quad g = \sum_{i=1}^{\infty} g(f_i) g_i, \quad \forall g \in X^*.$$

**Theorem 3.3** *Let  $X_d$  be an  $RCB$ -space and let  $\mathbb{G}$  be an  $X_d$ -frame for  $X$ . Then the following holds.*

- (a) *If  $\mathbb{G}$  has a dual  $X_d^*$ -frame, then all the dual  $X_d^*$ -frames of  $\mathbb{G}$  are precisely the sequences  $(L\delta_i)_{i=1}^\infty$ , where  $L : X_d \rightarrow X$  is a bounded linear extension of  $U_{\mathbb{G}}^{-1}$ .*
- (b) *If  $\mathbb{G}$  has an  $s$ -pseudo-dual, then all the  $s$ -pseudo-duals of  $\mathbb{G}$  are precisely the sequences  $(L\delta_i)_{i=1}^\infty$  where the operator  $L : \mathcal{D}(L)(\subseteq X_d) \rightarrow X$  has the properties:  $\mathcal{D}(L) \supseteq \text{lin}\{\delta_i\} \cup \mathcal{R}(U_{\mathbb{G}})$ ,  $L$  is a linear extension of  $U_{\mathbb{G}}^{-1}$  and  $L(\sum_{i=1}^n g_i(f) \delta_i) \rightarrow L(\sum_{i=1}^\infty g_i(f) \delta_i)$  as  $n \rightarrow \infty$  for every  $f \in X$ .*

**Proof.** First note that  $X$  is reflexive because of the assumptions.

(a) If  $L$  is a bounded linear extension of  $U_{\mathbb{G}}^{-1}$ , it is proved in [9, Prop. 3.4] that  $(L\delta_i)_{i=1}^\infty$  is a dual  $X_d^*$ -frame of  $\mathbb{G}$ . Let now  $\mathbb{F}$  be a dual  $X_d^*$ -frame of  $\mathbb{G}$ . By Theorem 3.1, the sequence  $\mathbb{F}$  has the form  $(T\delta_i)_{i=1}^\infty$  for some bounded surjective operator  $T : X_d \rightarrow X$ . Furthermore, (8) implies that  $T$  is an extension of  $U_{\mathbb{G}}^{-1}$ . Take  $L = T$ .

(b) is easy to prove using arguments from Theorem 3.1.  $\square$

Thus, when  $X_d$  is an  $RCB$ -space and  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ , then an  $s$ -pseudo-dual  $\mathbb{F}$  of  $\mathbb{G}$  is a dual  $X_d^*$ -frame of  $\mathbb{G}$  if and only if  $T_{\mathbb{F}}$  is bounded on  $\mathcal{D}(T_{\mathbb{F}})$ .

For conditions equivalent to the existence of dual  $X_d^*$ -frames of  $X_d$ -frames see [9, 29].

There exist series expansions in the form (8) where  $\mathbb{G}$  is an  $X_d$ -Bessel sequence and not an  $X_d$ -frame. As a trivial example, consider the Bessel sequence  $\mathbb{G} = (\frac{1}{i}e_i)_{i=1}^{\infty}$  and the expansions  $f = \sum_{i=1}^{\infty} \langle f, \frac{1}{i}e_i \rangle i e_i$ ,  $\forall f \in \mathcal{H}$ . Some characterizations from Theorems 3.1 and 3.3 connected to  $X_d$ -frames can easily be extended to  $X_d$ -Bessel sequences:

**Proposition 3.4** *Let  $X_d$  be an  $RCB$ -space and  $X$  be reflexive. Then the following statements hold.*

- (a) *The  $X_d$ -Bessel sequences  $\mathbb{G}$  for which there exists  $\mathbb{F}$  satisfying (8) are precisely the sequences  $(T\delta_i^*)_{i=1}^{\infty}$ , where  $T : X_d^* \rightarrow X^*$  is a bounded operator such that  $T^*$  has a left inverse  $L : \mathcal{D}(L)(\subseteq X_d) \rightarrow X$  satisfying  $\mathcal{D}(L) \supseteq \text{lin}\{\delta_i\} \cup \mathcal{R}(T^*)$  and  $L(\sum_{i=1}^n T\delta_i^*(f)\delta_i) \rightarrow L(\sum_{i=1}^{\infty} T\delta_i^*(f)\delta_i)$  as  $n \rightarrow \infty$  for every  $f \in X$ .*
- (b) *Let  $\mathbb{G}$  be an  $X_d$ -Bessel sequence for  $X$ . If there exists  $\mathbb{F}$  satisfying (8), then all possibilities for  $\mathbb{F}$  are precisely the sequences  $(T\delta_i)_{i=1}^{\infty}$  where the operator  $T : \mathcal{D}(T)(\subseteq X_d) \rightarrow X$  has the properties:  $\mathcal{D}(T) \supseteq \text{lin}\{\delta_i\} \cup \mathcal{R}(U_{\mathbb{G}})$ ,  $T$  is a left inverse of  $U_{\mathbb{G}}$  and  $T(\sum_{i=1}^n g_i(f)\delta_i) \rightarrow T(\sum_{i=1}^{\infty} g_i(f)\delta_i)$  as  $n \rightarrow \infty$  for every  $f \in X$ .*
- (c) *The sequences  $\mathbb{F}$  for which there exists an  $X_d$ -Bessel sequence  $\mathbb{G}$  for  $X$  satisfying (8) are precisely the sequences  $(T\delta_i)_{i=1}^{\infty}$ , where the operator  $T : \mathcal{D}(T)(\subseteq X_d) \rightarrow X$  satisfies the properties:  $\mathcal{D}(T) \supseteq \text{lin}\{\delta_i\}$ ,  $T(\sum_{i=1}^n c_i\delta_i) \rightarrow T(\sum_{i=1}^{\infty} c_i\delta_i)$  as  $n \rightarrow \infty$  for every  $\sum_{i=1}^{\infty} c_i\delta_i \in \mathcal{D}(T)$  and  $T$  has a bounded right inverse  $U : X \rightarrow X_d$  with  $\mathcal{R}(U) \subseteq \mathcal{D}(T)$ .*

Let  $X_d$  be an  $RCB$ -space. There is no need to characterize dual  $X_d^*$ -Bessel sequences of  $X_d$ -Bessel sequences, because if  $\mathbb{G}$  is an  $X_d$ -Bessel sequence for  $X$  and  $\mathbb{F}$  is an  $X_d^*$ -Bessel for  $X^*$  satisfying (8), then  $\mathbb{G}$  is an  $X_d$ -frame for  $X$  and  $\mathbb{F}$  is an  $X_d^*$ -frame for  $X^*$  [30].

## Connection between expansions in $X$ and $X^*$

Let  $X_d$  be an  $RCB$ -space and let  $\mathbb{G}$  be an  $X_d$ -Bessel sequence for  $X$ . If  $\mathbb{F}$  is an  $X_d^*$ -Bessel sequence for  $X^*$ , then  $\mathbb{F}$  satisfies (8) if and only if it satisfies

$$g = \sum_{i=1}^{\infty} g(f_i)g_i, \quad \forall g \in X^*, \quad (9)$$

see [30, Lemma 4.3]. If  $\mathbb{F}$  is not an  $X_d^*$ -Bessel sequence for  $X^*$ , such equivalence does not hold in general. There exist cases when both representations (8) and (9) hold (see Example 3.2) and there exist cases when (8) holds, but (9) does not hold (see Example 3.5 below). In Proposition 3.6 we determine a subset of  $X^*$  where the representation  $g = \sum_{i=1}^{\infty} g(f_i)g_i$  holds based on validity of (8). For investigation of the converse situation (conclusions for  $f = \sum g_i(f)f_i$  assuming validity of  $g = \sum_{i=1}^{\infty} g(f_i)g_i, \quad \forall g \in \mathcal{D}(U_{\mathbb{F}})$ ) see [27] switching the roles of  $(f_i)_{i=1}^{\infty}$  and  $(g_i)_{i=1}^{\infty}$ .

**Example 3.5** Consider the frame  $\mathbb{G} = (e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, \dots)$  for  $\mathcal{H}$  and the non-Bessel sequence  $\mathbb{F} = (e_1, e_1, -e_1, e_2, e_1, -e_1, e_3, e_1, -e_1, \dots)$ . Then (1) holds, but (2) does not hold - more precisely, the representation  $g = \sum_{i=1}^{\infty} \langle g, f_i \rangle g_i$  holds if and only if  $g$  belongs to the closed linear span of  $\{e_i\}_{i=2}^{\infty}$ .

**Proposition 3.6** Let  $X_d$  be an  $RCB$ -space,  $\mathbb{G}$  be an  $X_d$ -Bessel sequence for  $X$ , and  $\mathbb{F}$  satisfy (8). Then  $g = \sum_{i=1}^{\infty} g(f_i)g_i$  for every  $g \in D(T_{\mathbb{F}}^*)$ .

**Proof.** By Lemma 2.1, the sequence  $\mathbb{F}$  satisfies the lower  $X_d^*$ -frame condition. Consider the operator  $T_{\mathbb{F}}$  given by (6) and (7). Clearly,  $\mathbb{F} = (T_{\mathbb{F}}\delta_i)$  and  $T_{\mathbb{F}}$  is a left inverse of  $U_{\mathbb{G}}$ . Further, for every  $g \in D(T_{\mathbb{F}}^*)$  we have  $\sum_{i=1}^n g(f_i)g_i = T_{\mathbb{G}}(\sum_{i=1}^n gT_{\mathbb{F}}(\delta_i)\delta_i^*) \rightarrow U_{\mathbb{G}}^*T_{\mathbb{F}}^*(g) = g$  as  $n \rightarrow \infty$ .  $\square$

**Remark 3.7** The sequences in Example 3.5 fulfill the assumptions of Proposition 3.6,  $D(T_{\mathbb{F}}^*) \neq \mathcal{H}$ , and

$$g = \sum_{i=1}^{\infty} \langle g, f_i \rangle g_i \quad \text{if and only if} \quad g \in D(T_{\mathbb{F}}^*).$$

This shows that in general Proposition 3.6 can not be improved in the sense that the set of validity of  $g = \sum_{i=1}^{\infty} g(f_i)g_i$  can not be extended. However, there exist certain cases where the assumptions of Proposition 3.6 hold,  $D(T_{\mathbb{F}}^*) \neq X^*$ , and (9) holds - see Example 3.2 (note that  $E_1 \notin \mathcal{D}(T_{\mathbb{F}}^*)$ ).

## 4 Operators keeping the sequence type

For a given sequence  $\mathbb{G}$ , here we consider conditions on an operator  $V$  defined on  $\text{lin}\{g_i\}_{i=1}^\infty$  which preserve the sequence type. First consider the case of  $X_d$ -Bessel sequences. If  $\mathbb{G}$  is an  $X_d$ -Bessel sequence for  $X$ , then  $(Vg_i)_{i=1}^\infty$  is an  $X_d$ -Bessel sequence for  $X$  if and only if the operator  $VT_{\mathbb{G}}|_{\text{lin}\{\delta_i^*\}_{i=1}^\infty}$  is bounded. Note that the condition “the operator  $VT_{\mathbb{G}}|_{\text{lin}\{\delta_i^*\}_{i=1}^\infty}$  is bounded” can not be relaxed to the condition “the operator  $VT_{\mathbb{G}}|_{\text{lin}\{\delta_i^*\}_{i=1}^\infty}$  is bounded on the set  $\{\delta_i^*\}_{i=1}^\infty$ ”. Consider for example the Bessel sequence  $\mathbb{G} = (e_i)_{i=1}^\infty$  and the operator  $V$  given by  $Ve_i := e_1, i \in \mathbb{N}$ . In a similar way as in the  $X_d$ -Bessel sequence case, using Theorem 3.1 one can list conditions on  $V$  (more precisely, on  $T_{(Vg_i)}$ ) which are necessary and sufficient to preserve the type of the sequences discussed in Theorem 3.1 and we will skip the listing. In general, these conditions do not require  $V$  to be bounded. Consider for example the Bessel sequence  $\mathbb{G} = (\frac{1}{i}e_i)_{i=1}^\infty$  (or the sequence  $\mathbb{G} = (ie_i)_{i=1}^\infty$  satisfying the lower frame condition) and the operator  $V$  given by  $Ve_i := ie_i, i \in \mathbb{N}$ . However, operators which keep the  $X_d$ -Riesz basis property must be bounded:

**Lemma 4.1** *Let  $X_d$  be a CB-space and let  $\mathbb{F}$  be an  $X_d$ -Riesz basis for  $X$ . Let an operator  $V : \text{lin}\{f_i\}_{i=1}^\infty \rightarrow X$  be given. If  $(Vf_i)_{i=1}^\infty$  is an  $X_d$ -Riesz basis for  $X$ , then  $V$  is bounded.*

Below we concentrate on bounded operators and determine the additional conditions which are necessary and sufficient to preserve the sequence type. We are motivated by [1], where the author investigates constructions of frames using a given frame; this is of interest for construction of frames with desired properties suitable for applications.

**Proposition 4.2** *Let  $X$  be reflexive,  $X_d$  be an RCB-space and  $V$  be a bounded operator from  $X^*$  into  $X^*$  for (i)-(iv) and from  $X$  into  $X$  for (v). Then the following statements hold.*

- (i) *Let  $\mathbb{G}$  be an  $X_d$ -Bessel sequence for  $X$ . Then  $(Vg_i)_{i=1}^\infty$  is an  $X_d$ -Bessel sequence for  $X$ .*
- (ii) *Let  $\mathbb{G}$  be an  $X_d$ -frame for  $X$ . The sequence  $(Vg_i)_{i=1}^\infty$  is an  $X_d$ -frame for  $X$  if and only if  $V$  is surjective.*
- (iii) *Let  $\mathbb{G}$  be a Banach frame for  $X$  wrt  $X_d$ . The sequence  $(Vg_i)_{i=1}^\infty$  is a Banach frame for  $X$  wrt  $X_d$  if and only if  $V$  is surjective and it has a bounded right inverse  $W : X^* \rightarrow X^*$ .*

- (iv) Let  $\mathbb{G}$  be such that there exists an atomic decomposition  $(\mathbb{G}, \mathbb{F})$  for  $X$  wrt  $X_d$ . The sequence  $(Vg_i)_{i=1}^\infty$  is the first one in some atomic decomposition if and only if  $V$  is surjective and  $U_{\mathbb{G}}V^*$  has a left inverse  $L : \mathcal{D}(L)(\subseteq X_d) \rightarrow X$  satisfying  $\mathcal{D}(L) \supseteq \text{lin}\{\delta_i\} \cup \mathcal{R}(U_{\mathbb{G}}V^*)$  and  $L(\sum_{i=1}^n Vg_i(f)\delta_i) \rightarrow L(\sum_{i=1}^\infty Vg_i(f)\delta_i)$  as  $n \rightarrow \infty$  for every  $f \in X$ .
- (v) Let  $\mathbb{F}$  be an  $X_d$ -Riesz basis for  $X$ . The sequence  $(Vf_i)_{i=1}^\infty$  is an  $X_d$ -Riesz basis for  $X$  if and only if  $V$  is bijective.

**Proof.** We give sketch of the proofs.

(i) If  $B_{\mathbb{G}}$  denotes an  $X_d$ -Bessel bound for  $\mathbb{G}$ , then for every  $f \in X$  one has  $(Vg_i(f)) = (g_i(V^*f)) \in X_d$  and  $\|(Vg_i(f))\|_{X_d} \leq B_{\mathbb{G}}\|V\|\|f\|$ .

(ii) By (i),  $(Vg_i)_{i=1}^\infty$  is an  $X_d$ -Bessel sequence for  $X$ . Further, the sequence  $(Vg_i)_{i=1}^\infty$  satisfies the lower  $X_d$ -frame condition if and only if there exists  $\lambda > 0$  so that  $\|V^*f\| \geq \lambda\|f\|$ ,  $\forall f \in X$ , which is equivalent to  $V$  being surjective.

(iii) Let  $Q$  denote a Banach frame operator for  $\mathbb{G}$ .

First assume that  $V$  is surjective and  $V$  has a bounded right inverse  $W : X^* \rightarrow X^*$ . By (ii),  $\mathbb{G}$  is an  $X_d$ -frame for  $X$ . Further,  $W^*Q$  is a Banach frame operator for  $(Vg_i)_{i=1}^\infty$ .

Conversely, assume that  $(Vg_i)_{i=1}^\infty$  is a Banach frame for  $X$  wrt  $X_d$  and let  $Q_1$  denote a Banach frame operator for  $(Vg_i)_{i=1}^\infty$ . By (ii),  $V$  is surjective. Further,  $T_{\mathbb{G}}(Q_1)^*$  is a bounded right inverse of  $V$ .

(iv) and (v) follow easily using Theorem 3.1.  $\square$

**Remark** For a given  $X_d$ -frame (resp.  $X_d$ -Bessel sequence, Banach frame, atomic decomposition, sequence satisfying the lower  $X_d$ -frame condition)  $\mathbb{G}$  for  $X$ , not all the  $X_d$ -frames (resp.  $X_d$ -Bessel sequences, Banach frames, atomic decompositions, sequences satisfying the lower  $X_d$ -frame condition) for  $X$  can be obtained in the way  $(Vg_i)$  using an operator  $V$ . Consider for example the frame  $\mathbb{G} = (e_1, e_1, e_2, e_3, e_4, \dots)$  and the frame  $(e_1, e_2, e_3, e_4, e_5, \dots)$  for  $\mathcal{H}$ , which can not be written as  $(Vg_i)$  for any operator  $V$ . In contrary, when  $X_d$  is a  $CB$ -space and an  $X_d$ -Riesz basis  $\mathbb{F}$  for  $X$  is given, then every  $X_d$ -Riesz basis  $\mathbb{W} = (w_i)$  for  $X$  can be written in the way  $(Vf_i)$  using the operator  $V = T_{\mathbb{W}}T_{\mathbb{F}}^{-1}$  which is a bounded bijection of  $X$  onto  $X$ .

## 5 Application to Hilbert frame theory

### Characterization of the $s$ -pseudo-duals of a frame

As a consequence of Theorem 3.3, the  $s$ -pseudo-duals of a (Hilbert) frame can be characterized as follows.

**Corollary 5.1** *Let  $\mathbb{G}$  be an overcomplete frame for  $\mathcal{H}$ . Then the  $s$ -pseudo-duals of  $\mathbb{G}$  are precisely the sequences  $\{L\delta_i\}_{i=1}^\infty$  where the operator  $L : \mathcal{D}(L)(\subseteq \ell^2) \rightarrow \mathcal{H}$  has the properties:  $\mathcal{D}(L) \supseteq \text{lin}\{\delta_i\} \cup \mathcal{R}(U_{\mathbb{G}})$ ,  $L$  is an extension of  $U_{\mathbb{G}}^{-1}$  and  $L(\sum_{i=1}^n \langle f, g_i \rangle \delta_i) \rightarrow L(\sum_{i=1}^\infty \langle f, g_i \rangle \delta_i)$  as  $n \rightarrow \infty$  for every  $f \in X$ .*

An  $s$ -pseudo-dual of  $\mathbb{G}$  is a frame if and only if  $L$  is bounded on  $\mathcal{D}(L)$ .

### Class of frames $\mathbb{G}$ such that any sequence $\mathbb{F}$ satisfying (1) is automatically a dual frame of $\mathbb{G}$

There has been a question in the frame theory to characterize frames whose  $s$ -pseudo-duals are automatically frames. As a consequence of Corollary 5.1, the following statement gives a class of such frames.

**Proposition 5.2** *Let  $\mathbb{G} \in \mathcal{H}^{\mathbb{N}}$  be a frame for  $\mathcal{H}$  and let  $(\mathcal{R}(U_{\mathbb{G}}))^\perp$  be finite-dimensional. Then any  $s$ -pseudo-dual of  $\mathbb{G}$  is a dual frame of  $\mathbb{G}$ .*

As an illustration of Proposition 5.2, consider the frame  $\mathbb{G} = (e_1, e_1, e_2, e_3, e_4, \dots)$  for  $\mathcal{H}$ . Any  $s$ -pseudo-dual of  $\mathbb{G}$  has the structure  $(w, e_1 - w, e_2, e_3, e_4, \dots)$  for some  $w \in \mathcal{H}$  and it is a frame for  $\mathcal{H}$ .

### The lower frame condition and its connection to expansions

It is proved in [5] that a sequence  $\mathbb{F}$  satisfies the lower frame condition if and only if there exists a Bessel sequence  $\mathbb{G}$  such that  $f = \sum_{i=1}^\infty \langle f, f_i \rangle g_i$ ,  $\forall f \in \mathcal{D}(U_{\mathbb{F}})$ . Here we investigate the situation when  $\mathbb{F}$  is the synthesis sequence in the expansion.

**Proposition 5.3** *Let  $\mathbb{F} \in \mathcal{H}^{\mathbb{N}}$  satisfy the lower frame condition and let  $U_{\mathbb{F}}$  be densely defined. Then there exists a Bessel sequence  $\mathbb{G}$  in  $\mathcal{H}$  satisfying*

$$f = \sum_{i=1}^\infty \langle f, g_i \rangle f_i \text{ for every } f \in \mathcal{H} \text{ such that } \sum_{i=1}^\infty \langle f, g_i \rangle f_i \text{ converges.} \quad (10)$$

**Proof.** Since  $\mathcal{R}(U_{\mathbb{F}})$  is a closed subspace of  $\ell^2$  [5] and  $U_{\mathbb{F}}^{-1}$  is bounded, there exists a bounded extension  $V : \ell^2 \rightarrow \mathcal{H}$  of  $U_{\mathbb{F}}^{-1}$ . Define  $g_i = V\delta_i, i \in \mathbb{N}$ . Clearly,  $\mathbb{G}$  is a Bessel sequence in  $\mathcal{H}$ . Let  $f \in \mathcal{H}$  be such that  $\sum_{i=1}^{\infty} \langle f, g_i \rangle f_i$  converges. Using Theorem 3.1(ii) and the fact that  $T_{\mathbb{F}} \subseteq U_{\mathbb{F}}^*$  [7, Prop. 4.6], we have

$$\sum_{i=1}^n \langle f, g_i \rangle f_i \rightarrow T_{\mathbb{F}} \left( \sum_{i=1}^{\infty} \langle f, g_i \rangle \delta_i \right) = U_{\mathbb{F}}^* V^*(f) = f. \quad \square$$

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